Ex: A free fall ball. position function $y(t)$.

initial position $y(0)=10$ — ①

initial velocity $y'(0)=0$ — ②

physics: $y''=-gt$ (*)

$\rightsquigarrow$ general soln for (*):

$y=-\frac{g}{2}t^2+C_1t+C_2$. — (†)

plug in (†) to ① & ②

$y(0)=C_2=10$

$y'=-gt+C_1$

so $y'(0)=C_1=0$

Unique soln to the IVP: (*)+①, ②

$y=\frac{-g}{2}t^2+10$ □

Prop If y_1, y_2 are solns to a homogeneous linear ODE. then any linear combination of y_1, y_2 is also a soln $C_1y_1+C_2y_2$.

pf: (2nd order).

$y''+a_1(x)y'+a_0(x)y=0$ (*)

y_1, y_2 are solutions to (*) so.

$k(y_1''+a_1y_1'+a_0y_1)=0$ — ①

$y_2''+a_1y_2'+a_0y_2=0$ — ②

① × k ← constant scalar.

$(ky_1)''+a_1(ky_1)'+a_0ky_1=0 \Rightarrow ky_1$ is also a soln

① ⊕ ②:

$y_1''+y_2''+a_1y_1'+a_1y_2'+a_0y_1+a_0y_2=0$

$\to (y_1+y_2)''+a_1(y_1+y_2)'+a_0(y_1+y_2)=0$

$\Rightarrow$ $y_1 + y_2$ is another soln to (*)

Eg. : $y'' - 3y' + 2y = 0$ —— (*)

Try $y = e^{kx}$ $y' = ky$ $y'' = k^2 y$

$y'' - 3y' + 2y = (k^2 - 3k + 2)\, y = 0$ if y is a soln.

if, $k = 1, 2$, this indeed vanishes

so $y_1 = e^x$, $y_2 = e^{2x}$ are all solns to (*)

$\leadsto$ Prop $y = C_1 e^x + C_2 e^{2x}$ (†) is also a solution

C_1, C_2 are arbitrary constants

but (*) is of 2nd order, so a general soln has two arbitrary constants

already has 2 arbitrary constants so this is the general solution to (*)

Q 2b. $y(x) = \int_{1-10x}^{1} \frac{u^3}{1+u^2}\, du$. Find y'.

Fundamental thm of Calculus: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$y = -\int_1^{1-10x} \frac{u^3}{1+u^2}\, du$.

let $v = 1 - 10x$ $\frac{dv}{dx} = -10$

then $\frac{d}{dv}\int_1^v \frac{u^3}{1+u^2}\, du = \frac{v^3}{1+v^2}$

(fund thm of Calculus)

$y' = -\frac{d}{dx}\int_1^v \frac{u^3}{1+u^2}\, du$

$= -\frac{dv}{dx}\frac{d}{dv}\int_1^v \frac{u^3}{1+u^2}\, du = -(-10)\frac{v^3}{1+v^2} = 10\,\frac{(1-10x)^3}{1+(1-10x)^2}$ □

~~4.1~~ 5 General structures of linear ODEs (optiopnal)

Fact: A general solution to a n-th order ODE typically involve n indeterminate constants.

Example 4.3. A falling ball: $y'' = -g$ (gravitational constant). Initial conditions" initial position and velocity.

↑ position of the ball (vertical direction as the y-axis) regarded as a function of time, t

$$\frac{d}{dt}(y') = -g$$

$$y' = -gt + C_1$$

$$y = -\frac{g}{2}t^2 + C_1 t + C_2 \leftarrow \text{general soln}$$

IVP: given that $y(0) = 10$ ft (initial position)

$y'(0) = 0$ (initial velocity)

Proposition 1 (structure of homogeneous linear ODEs). If y_1, y_2 are two solutions of a homogeneous ODE, then for any constants C_1, C_2, $y = C_1 y_1 + C_2 y_2$ is also a solution.

e.g. $y'' + a_1(x) y' + a_0(x) y = 0$ homogeneous 2nd order ODE.

Example 4.4. Find all solutions of the ODE: $y'' - 3y' + 2y = 0$.

Proposition 2 (structure of linear ODEs). A general solution y to a linear ODE has the form:

$$y = y_h + y_p,$$

where y_h is the general solution to the linear ODE's associated homogeneous linear ODE; y_p is a "particular solution" to the ODE itself.

the ODE with the same homogeneous part but inhomogeneous term replaced by 0

Example 4.5. Find all solutions of the ODE: (#) $y'' - 3y' + 2y = 2.$ ← inhomogeneous term.

the associated homogeneous linear ODE

$$y'' - 3y' + 2y = 0$$

We found that

$$y_h = C_1 e^x + C_2 e^{2x}$$

So according to Prop 2, a general soln to (#)

is $y_g = y_h + y_p$ $\quad y_p = 1$

$$= C_1 e^x + C_2 e^{2x} + 1$$

□

pf of Prop 2 (2nd order)

$$y'' + a_1 y' + a_0 y = g \qquad (1)$$

y_h. satisfied the associated homogeneous equation

$$y_h'' + a_1 y_h' + a_0 y_h = 0 \qquad (2)$$

(1) - (2) : $\quad y'' - y_h'' + a_1 y' - a_1 y_h' + a_0 y - a_1 y_h = g$

$\rightsquigarrow \quad (y - y_h)'' + a_1 (y - y_h)' + a_0 (y - y_h) = g.$

$\rightsquigarrow \quad y - y_h$ is a solution to (1)

pick one, say y_p

$\Rightarrow \quad y - y_h = y_p$

$$y = y_h + y_p. \qquad \square$$